

AN EFFICIENT FILTERED SCHEME FOR SOME FIRST ORDER HAMILTON-JACOBI-BELLMAN EQUATIONS*

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Abstract. We introduce a new class of “filtered” schemes for some first order non-linear Hamilton-Jacobi-Bellman equations. The work follows recent ideas of Froese and Oberman (SIAM J. Numer. Anal., Vol 51, pp.423-444, 2013). The proposed schemes are not monotone but still satisfy some ϵ -monotone property. Convergence results and precise error estimates are given, of the order of $\sqrt{\Delta x}$ where Δx is the mesh size. The framework allows to construct finite difference discretizations that are easy to implement, high-order in the domains where the solution is smooth, and provably convergent, together with error estimates. Numerical tests on several examples are given to validate the approach, also showing how the filtered technique can be applied to stabilize an otherwise unstable high-order scheme.

Key words. Hamilton-Jacobi equation, high-order schemes, ϵ -monotone scheme, viscosity solutions, error estimates

1. Introduction. In this work, our aim is to develop high-order and convergent schemes for first order Hamilton-Jacobi (HJ) equations of the following form

$$\partial_t v + H(x, \nabla v) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d \quad (1.1)$$

$$v(0, x) = v_0(x), \quad x \in \mathbb{R}^d. \quad (1.2)$$

Basic assumptions on the Hamiltonian H and the initial data v_0 will be introduced in the next section. It is well known that, in the one dimensional case, there is a strong link between Hamilton-Jacobi equations and scalar conservation laws. Namely, the viscosity solution of the evolutive HJ equation is the primitive of the entropy solution of the corresponding hyperbolic conservation law with the same hamiltonian. There are several schemes developed for hyperbolic conservation law (see references [15] [16], [7], [13]). Most of the numerical ideas to solve hyperbolic conservation law can be extended to HJ equations. Well known high-order essentially non-oscillatory (ENO) scheme have been introduced by A. Harten et al. in [17] for hyperbolic conservation laws, and then extended to HJ equation by Osher and Shu [20]. ENO schemes have shown to have high-order accuracy although a precise convergence result is still missing and, for this property, they have been quite successful in many applications. Despite the fact that there is no convergence proof of ENO schemes towards the viscosity solution of (1.1) in the general case, convergence results may hold for related schemes, e.g. MUSCL schemes, as it has been proved by Lions and Souganidis in [19]. Convergence results of some non monotone scheme have also been shown in particular cases [5]. Another interesting result has been proved by Fjordholm et al. [11], they have shown that ENO interpolation is stable but the stability result is not sufficient to conclude total variation boundedness (TVB) of the ENO reconstruction procedure. In [10], a conjecture related to weak total variation property for ENO schemes is given. Finally, let us also mention [6], where weighted essentially non-oscillatory (WENO) schemes have been applied to HJ equations; the convergence

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proof of the scheme relies also on the work of Ferretti [9] where higher than first order schemes are proposed in a semi-Lagrangian setting, yet with restrictive conditions on the mesh steps.

In this paper we give a very simple way to construct high-order schemes in a convergent framework. It is known (by Godunov's theorem) that a monotone scheme can be at most first order. Therefore it is needed to look for non-monotone schemes. The difficulty is then to combine non-monotonicity of the scheme and convergence towards the viscosity solution of (1.1), and also to obtain error estimates. In our approach we will adapt a general idea of Froese and Oberman [12], that was presented for stationary second order Hamilton-Jacobi equations and based on the use of a “filter” function. Here we focus mainly on the case of evolutive first order Hamilton-Jacobi equation (1.1), and an adaptation to the steady case will be also considered. We will design a slightly different filter function for which the filtered scheme is still an “ ϵ -monotone” scheme (see Eq.2.12), but that improves the numerical results. Let us mention also the work [4] for steady equations where some ϵ -monotone semi-Lagrangian schemes are studied.

The paper is organized as follows. In Section 2, we present the schemes and give main convergence results. Section 3 is devoted to the numerical tests on several academic examples to illustrate our approach in one and two-dimensional cases. A test on nonlinear steady equations, as well as an evolutive “obstacle” HJ equation in the form of $\min(u_t + H(x, u_x), u - g(x)) = 0$ for a given function g are also included in this section. Finally, Section 4 contains our concluding remarks.

2. Definitions and main results.

2.1. Setting of the problem. Let us denote by $|\cdot|$ the Euclidean norm on \mathbb{R}^d ($d \geq 1$). The following classical assumptions will be considered in the sequel of this paper:

(A1) v_0 is Lipschitz continuous function i.e. there exist $L_0 > 0$ such that for every $x, y \in \mathbb{R}^d$,

$$|v_0(x) - v_0(y)| \leq L_0|x - y|. \quad (2.1)$$

(A2) $H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies, for some constant $C \geq 0$, for all $p, q, x, y \in \mathbb{R}^d$:

$$|H(y, p) - H(x, p)| \leq C(1 + |p|)|y - x|, \quad (2.2)$$

and

$$|H(x, q) - H(x, p)| \leq C(1 + |x|)|q - p|. \quad (2.3)$$

Under assumptions (A1) and (A2) there exists a unique viscosity solution for (1.1) (see Ishii [18]). Furthermore v is locally Lipschitz continuous on $[0, T] \times \mathbb{R}^d$.

For clarity of presentation we focus on the one-dimensional case and consider the following simplified problem:

$$v_t + H(x, v_x) = 0, \quad x \in \mathbb{R}, \quad t \in [0, T], \quad (2.4)$$

$$v(0, x) = v_0(x), \quad x \in \mathbb{R}. \quad (2.5)$$

2.2. Construction of the filtered scheme. Let $\tau = \Delta t > 0$ be a time step (in the form of $\tau = \frac{T}{N}$ for some $N \geq 1$), and $\Delta x > 0$ be a space step. A uniform mesh in time is defined by $t_n := n\tau$, $n \in [0, \dots, N]$, and in space by the nodes $x_j := j\Delta x$, $j \in \mathbb{Z}$.

The construction of a filtered scheme needs three ingredients:

- a monotone scheme, denoted S^M
- a high-order scheme, denoted S^A
- a bounded “filter” function, $F : \mathbb{R} \rightarrow \mathbb{R}$.

The high-order scheme need not be convergent nor stable; the letter A stands for “arbitrary order”, following [12]. For a start, S^M will be based on a finite difference scheme. Later on we will also propose a definition of S^M based on a semi-Lagrangian scheme.

Then, the filtered scheme S^F is defined as

$$u_j^{n+1} \equiv S^F(u^n)_j := S^M(u^n)_j + \epsilon \tau F\left(\frac{S^A(u^n)_j - S^M(u^n)_j}{\epsilon \tau}\right), \quad (2.6)$$

where $\epsilon = \epsilon_{\tau, \Delta x} > 0$ is a parameter satisfying

$$\lim_{(\tau, \Delta x) \rightarrow 0} \epsilon = 0. \quad (2.7)$$

More hints on the choice of ϵ will be given later on.

The scheme is initialized in the standard way, i.e.

$$u_j^0 := v_0(x_j), \quad \forall j \in \mathbb{Z}. \quad (2.8)$$

Now we make precise some requirements on S^M , S^A and the function F .

Definition of the monotone finite difference scheme S^M

Following Crandall and Lions [7], we consider a finite difference scheme written as $u^{n+1} = S^M(u^n)$ with

$$S^M(u^n)(x) := u^n(x) - \tau h^M(x, D^- u^n(x), D^+ u^n(x)), \quad (2.9)$$

with

$$D^\pm u(x) := \pm \frac{u(x \pm \Delta x) - u(x)}{\Delta x},$$

where h^M corresponds to a monotone numerical Hamiltonian that will be made precise below. We will denote also $S^M(u^n)_j := S^M(u^n)(x_j)$. Therefore the scheme also reads, for all $j \in \mathbb{Z}$, $\forall n \geq 0$:

$$u_j^{n+1} := u_j^n - \tau h^M(x_j, D^- u_j^n, D^+ u_j^n), \quad D^\pm u_j^n := \pm \frac{u_{j\pm 1}^n - u_j^n}{\Delta x}. \quad (2.10)$$

(A3) Assumptions on S^M

We will use the following assumptions throughout this paper:

- (i) h^M is a Lipschitz continuous function.
- (ii) (*consistency*) $\forall x, \forall u, h^M(x, u, u) = H(x, u)$.
- (iii) (*monotonicity*) for any functions u, v ,
 $u \leq v \implies S^M(u) \leq S^M(v)$.

In practice condition (A3)-(iii) is only required at mesh points and the condition reads

$$u_j \leq v_j, \quad \forall j, \quad \implies \quad S^M(u)_j \leq S^M(v)_j, \quad \forall j. \quad (2.11)$$

At this stage, we notice that under condition (A3) the filtered scheme is “ ϵ -monotone” in the sense that

$$u_j \leq v_j, \quad \forall j, \quad \implies \quad S^M(u)_j \leq S^M(v)_j + \epsilon \tau, \quad \forall j. \quad (2.12)$$

with $\epsilon \rightarrow 0$ as $(\tau, \Delta x) \rightarrow 0$. This implies the convergence of the scheme by Barles-Souganidis convergence theorem (see [2, 1]).

REMARK 2.1. Under assumption (i), the consistency property (ii) is equivalent to say that, for any $v \in C^2([0, T] \times \mathbb{R})$, there exists a constant $C_M \geq 0$ independant of Δx such that

$$\left| h^M(x, D^-v(x), D^+v(x)) - H(x, v_x) \right| \leq C_M \Delta x \|\partial_{xx}v\|_\infty. \quad (2.13)$$

The same statement holds true if (2.13) is replaced by the following consistency error estimate:

$$\begin{aligned} \mathcal{E}_{SM}(v)(t, x) &:= \left| \frac{v(t + \tau, x) - S^M(v(t, \cdot))(x)}{\tau} - (v_t(t, x) + H(x, v_x(t, x))) \right| \\ &\leq C_M \left(\tau \|\partial_{tt}v\|_\infty + \Delta x \|\partial_{xx}v\|_\infty \right). \end{aligned} \quad (2.14)$$

REMARK 2.2. Assuming (i), it is easily shown that the monotonicity property (iii) is equivalent to say that $h^M = h^M(x, u^-, u^+)$ satisfies, a.e. $(x, u^-, u^+) \in \mathbb{R}^3$:

$$\frac{\partial h^M}{\partial u^-} \geq 0, \quad \frac{\partial h^M}{\partial u^+} \leq 0, \quad (2.15)$$

and the CFL condition

$$\frac{\tau}{\Delta x} \left(\frac{\partial h^M}{\partial u^-}(x, u^-, u^+) - \frac{\partial h^M}{\partial u^+}(x, u^-, u^+) \right) \leq 1. \quad (2.16)$$

When using finite difference schemes, it is assumed that the CFL condition (2.16) is satisfied, and that can be written equivalently in the form

$$c_0 \frac{\tau}{\Delta x} \leq 1, \quad (2.17)$$

where c_0 is a constant independant of τ and Δx .

EXAMPLE 2.1. Let us consider the Lax-Friedrichs numerical Hamiltonian is

$$h^{M, LF}(x, u^-, u^+) := H(x, \frac{u^- + u^+}{2}) - \frac{c_0}{2}(u^+ - u^-)$$

where $c_0 > 0$ is a constant. The scheme is consistent; it is furthermore monotone under the conditions $\max_{x,p} |\partial_p H(x, p)| \leq c_0$, and $c_0 \frac{\tau}{\Delta x} \leq 1$.

Definition of the high-order scheme S^A : we consider an iterative scheme of “high-order” in the form $u^{n+1} = S^A(u^n)$, written as

$$S^A(u^n)(x) = u^n(x) - \tau h^A(x, D^{k,-}u^n(x), \dots, D^-u^n(x), D^+u^n(x), \dots, D^{k,+}u^n(x)),$$

where h^A corresponds to a “high-order” numerical Hamiltonian, and $D^{\ell, \pm}u(x) := \pm \frac{u(x \pm \ell \Delta x) - u(x)}{\Delta x}$ for $\ell = 1, \dots, k$. To simplify the notations we may write (2.18) in the more compact form

$$S^A(u^n)(x) = u^n(x) - \tau h^A(x, D^\pm u^n(x)) \quad (2.18)$$

even if there is a dependency on ℓ in $(D^{\ell, \pm}u^n(x))_{\ell=1, \dots, k}$.

(A4) Assumptions on S^A :

We will use the following assumptions:

- (i) h^A is a Lipschitz continuous function.
- (ii) (high-order consistency) There exists $k \geq 2$, for all $\ell \in [1, \dots, k]$, for any function $v = v(t, x)$ of class $C^{\ell+1}$, there exists $C_{A,\ell} \geq 0$,

$$\mathcal{E}_{S^A}(v)(t, x) := \left| \frac{v(t + \tau, x) - S^A(v(t, \cdot))(x)}{\tau} - (v_t(t, x) + H(x, v_x(t, x))) \right| \quad (2.19)$$

$$\leq C_{A,\ell} \left(\tau^\ell \|\partial_t^{\ell+1} v\|_\infty + \Delta x^\ell \|\partial_x^{\ell+1} v\|_\infty \right). \quad (2.20)$$

Here v_x^ℓ denotes the ℓ -th derivative of v w.r.t. x .

REMARK 2.3. The high-order consistency implies, for all $\ell \in [1, \dots, k]$, and for $v \in C^{\ell+1}(\mathbb{R})$,

$$\left| h^A(x, \dots, D^-v, D^+v, \dots) - H(x, v_x) \right| \leq C_{A,\ell} \|\partial_x^{\ell+1} v\|_\infty \Delta x^\ell.$$

EXAMPLE 2.2. (Centered scheme) A typical example with $k = 2$ is obtained with the centered TVD (Total Variation Diminishing) approximation in space and the Runge-Kutta 2nd order scheme in time (or Heun scheme):

$$S_0(u^n)_j := u_j^n - \tau H(x_j, \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}), \quad (2.21a)$$

and

$$S^A(u) := \frac{1}{2}(u + S_0(S_0(u))). \quad (2.21b)$$

Of course there is no reason for the centered scheme to be stable (as it will be shown in the numerical section). Using a filter will help stabilize the scheme. A similar example with $k = 3$ can be obtained with any third order finite difference approximation in space and the TVD-RK3 scheme in time [14].

Definition of the filter function F . We recall that Froese and Oberman's filter function used in [12] is:

$$\tilde{F}(x) = \text{sign}(x) \max(1 - ||x| - 1|, 0) = \begin{cases} x & |x| \leq 1. \\ 0 & |x| \geq 2. \\ -x + 2 & 1 \leq x \leq 2. \\ -x - 2 & -2 \leq x \leq -1. \end{cases}$$

In the present work we define a new filter function simply as follows:

$$F(x) := x 1_{|x| \leq 1} = \begin{cases} x & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.22)$$

The idea of the present filter function is to keep the high-order scheme when $|h^A - h^M| \leq \epsilon$ (because then $|S^A - S^M|/(\tau\epsilon) \leq 1$ and $S^F = S^M + \tau\epsilon F(\frac{S^A - S^M}{\tau\epsilon}) \equiv S^A$), whereas $F = 0$ and $S^F = S^M$ if that bound is not satisfied, i.e., the scheme is simply given by the monotone scheme itself. Clearly the main difference is the discontinuity at $x = -1, 1$.

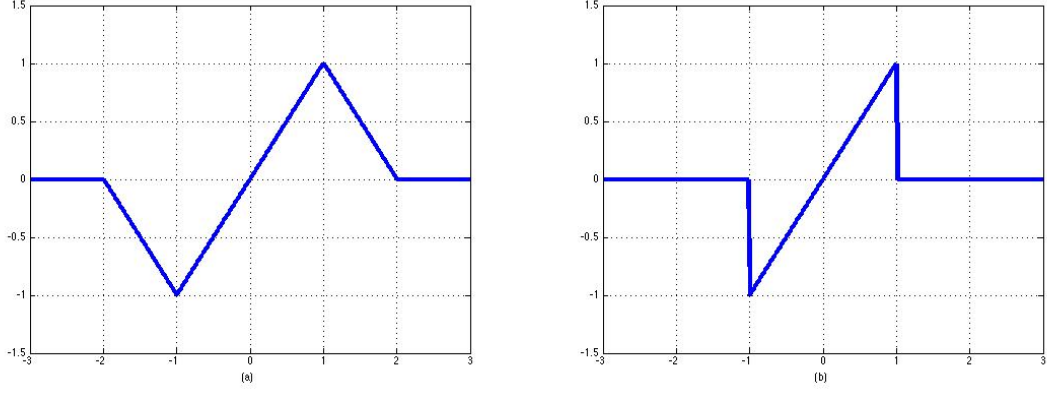


FIG. 2.1. Froese and Oberman's filter (left), new filter (right)

2.3. Convergence result. The following theorem gives several basic convergence results for the filtered scheme. Note that the high-order assumption (A4) will not be necessary to get the error estimates (i)-(ii). It will be only used to get a high-order consistency error estimate in the regular case (part (iii)). Globally the scheme will have just an $O(\sqrt{\Delta x})$ rate of convergence for just Lipschitz continuous solutions because the jumps in the gradient prevent high-order accuracy on the kinks.

THEOREM 2.1. *Assume (A1)-(A2), and let v_0 be bounded. We assume also that S^M satisfies (A3), and $|F| \leq 1$. Let u^n denote the filtered scheme (2.6). Let $v_j^n := v(t_n, x_j)$ where v is the exact solution of (2.4). Assume*

$$0 < \epsilon \leq c_0 \sqrt{\Delta x} \quad (2.23)$$

for some constant $c_0 > 0$.

(i) *The scheme u^n satisfies the Crandall-Lions estimate*

$$\|u^n - v^n\|_\infty \leq C \sqrt{\Delta x}, \quad \forall n = 0, \dots, N. \quad (2.24)$$

for some constant C independent of Δx .

(ii) *(First order convergence for classical solutions.) If furthermore the exact solution v belongs to $C^2([0, T] \times \mathbb{R})$, and $\epsilon \leq c_0 \Delta x$ (instead of (2.23)), then, we have*

$$\|u^n - v^n\|_\infty \leq C \Delta x, \quad n = 0, \dots, N, \quad (2.25)$$

for some constant C independent of Δx .

(iii) *(Local high-order consistency.) Assume that S^A is a high-order scheme satisfying (A4) for some $k \geq 2$. Let $1 \leq \ell \leq k$ and v be a $C^{\ell+1}$ function in a neighborhood of a point $(t, x) \in (0, T) \times \mathbb{R}$. Assume that*

$$(C_{A,1} + C_M) \left(\|v_{tt}\|_\infty \tau + \|v_{xx}\|_\infty \Delta x \right) \leq \epsilon. \quad (2.26)$$

Then, for sufficiently small $t_n - t$, $x_j - x$, τ , Δx , it holds

$$S^F(v^n)_j = S^A(v^n)_j$$

and, in particular, a local high-order consistency error for the filtered scheme S^F holds:

$$\mathcal{E}_{S^F}(v^n)_j \equiv \mathcal{E}_{S^A}(v^n)_j = O(\Delta x^\ell)$$

(the consistency error \mathcal{E}_{S^A} is defined in (2.19)).

Proof. (i) Let $w_j^{n+1} = S^M(w^n)_j$ be defined with the monotone scheme only, with $w_j^0 = v_0(x_j) = u_j^0$. By definitions,

$$u_j^{n+1} - w_j^{n+1} = S^M(u^n)_j - S^M(w^n)_j + \epsilon \tau F(\cdot)$$

Hence, by using the monotonicity of S^M ,

$$\max_j |u_j^{n+1} - w_j^{n+1}| \leq \max_j |u_j^n - w_j^n| + \epsilon \tau,$$

and by recursion, for $n \leq N$,

$$\max_j |u_j^n - w_j^n| \leq \epsilon n \tau \leq T \epsilon.$$

On the other hand, by Crandall and Lions [7], an error estimate holds for the monotone scheme:

$$\max_j |w_j^n - v_j^n| \leq C \sqrt{\Delta x},$$

for some $C \geq 0$. By summing up the previous bounds, we deduce

$$\max_j |u_j^n - v_j^n| \leq C \sqrt{\Delta x} + T \epsilon,$$

and together with the assumption on ϵ , it gives the desired result.

(ii) Let $\mathcal{E}_j^n := \frac{v_j^{n+1} - S^M(v^n)_j}{\tau}$. If the solution is C^2 regular with bounded second order derivatives, then the consistency error is bounded by

$$|\mathcal{E}_j^n| \leq C_M(\tau + \Delta x). \quad (2.27)$$

Hence

$$\begin{aligned} |u_j^{n+1} - v_j^{n+1}| &= |S^M(u^n)_j - S^M(v^n)_j + \tau \mathcal{E}_j^n + \tau \epsilon F(\cdot)| \\ &\leq \|u^n - v^n\|_\infty + \tau \|\mathcal{E}^n\|_\infty + \tau \epsilon. \end{aligned}$$

By recursion, for $n\tau \leq T$,

$$\|u^n - v^n\|_\infty \leq \|u^0 - v^0\|_\infty + T \left(\max_{0 \leq k \leq N-1} \|\mathcal{E}^k\|_\infty + \epsilon \right).$$

Finally by using the assumption on ϵ , the bound (2.27) and the fact that $\tau = O(\Delta x)$ (using CFL condition (2.17)), we get the desired result.

(iii) To prove that $S^F(v^n)_j = S^A(v^n)_j$, one has to check that

$$\frac{|S^A(v^n)_j - S^M(v^n)_j|}{\epsilon \tau} \leq 1$$

as $(\tau, \Delta x) \rightarrow 0$. By using the consistency error definitions,

$$\begin{aligned} \frac{|S^A(v^n)_j - S^M(v^n)_j|}{\tau} &= \left| \frac{v_j^{n+1} - S^M(v^n)_j}{\tau} + v_t(t_n, x_j) + H(x_j, v_x(t_n, x_j)) \right. \\ &\quad \left. - \left(\frac{v_j^{n+1} - S^A(v^n)_j}{\tau} + v_t(t_n, x_j) + H(x_j, v_x(t_n, x_j)) \right) \right| \\ &\leq |\mathcal{E}_{SM}(v^n)_j| + |\mathcal{E}_{SA}(v^n)_j| \\ &\leq (C_{A,1} + C_M)(\tau \|v_{tt}\|_\infty + \Delta x \|v_{xx}\|_\infty) \end{aligned}$$

Hence the desired result follows. \square

REMARK 2.4. Other approaches. *It is already known from the original work of Osher and Shu [20] that it is possible to modify an ENO scheme in order to obtain a convergent scheme. For instance, if $D^{\pm, A}u_j^n$ denotes a high-order finite difference derivative estimate (of ENO type), a projection on the first order finite difference derivative $D^\pm u_j^n$ can be used, up to a controlled error (see in particular Remark 2.2 of [20]):*

$$\text{instead of } D^{\pm, A}u_j^n, \quad \text{use } P_{[D^\pm u_j^n, M\Delta x]}(D^{\pm, A}u_j^n)$$

where $P_{[a,b]}(y)$ is the projection defined by:

$$P_{[a,b]}(y) := \begin{cases} y & \text{if } a - b \leq y \leq a + b \\ a - b & \text{if } y \leq a - b \\ a + b & \text{if } y \geq a + b \end{cases}$$

and $M > 0$ is some constant greater than the expected value $\frac{1}{2}|u_{xx}(t_n, x_j)|$. However, we emphasize that in our approach we do not consider a projection but a perturbation with a filter, which is slightly different. Indeed, by using a projection into an interval of the form $[a - M\Delta x, a + M\Delta x]$ where $a = D^\pm u_i^n$, numerical tests show that we may choose too often one of the extremal values $a \pm M\Delta x$ which is then produces an overall too big error (worse than using the first order finite differences).

Following the present approach, we would rather advice to use,

$$\text{instead of } D^{\pm, A}u_j^n, \quad \text{the value } \tilde{P}_{[D^\pm u_j^n, M\Delta x]}(D^{\pm, A}u_j^n)$$

where $\tilde{P}_{[a,b]}(y)$ is defined by:

$$\tilde{P}_{[a,b]}(y) := \begin{cases} y & \text{if } a - b \leq y \leq a + b \\ a & \text{if } y \notin [a - b, a + b] \end{cases}$$

REMARK 2.5. Filtered semi-Lagrangian scheme. *Let us consider the case of*

$$H(x, p) := \min_{b \in B} \max_{a \in A} \{-f(x, a, b) \cdot p - \ell(x, a, b)\}, \quad (2.28)$$

where $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ are non-empty compact sets (with $m, n \geq 1$), $f : \mathbb{R}^d \times A \times B \rightarrow \mathbb{R}^d$ and $\ell : \mathbb{R}^d \times A \times B \rightarrow \mathbb{R}$ are Lipschitz continuous w.r.t. x : $\exists L \geq 0$, $\forall (a, b) \in A \times B$, $\forall x, y$:

$$\max(|f(x, a, b) - f(y, a, b)|, |\ell(x, a, b) - \ell(y, a, b)|) \leq L|x - y|. \quad (2.29)$$

(We notice that (A2) is satisfied for hamiltonian functions such as (2.28).) Let $[u]$ denote the P^1 -interpolation of u in dimension one on the mesh (x_j) , i.e.

$$x \in [x_j, x_{j+1}] \quad \Rightarrow \quad [u](x) := \frac{x_{j+1} - x}{\Delta x} u_j + \frac{x - x_j}{\Delta x} u_{j+1}. \quad (2.30)$$

Then a monotone SL scheme can be defined as follows:

$$S^M(u^n)_j := \min_{a \in A} \max_{b \in B} \left([u^n](x_j + \tau f(x_j, a, b)) + \tau \ell(x_j, a, b) \right). \quad (2.31)$$

A filtered scheme based on SL can then be defined by (2.6) and (2.31). Convergence result as well as error estimates could also be obtained in this framework. (For error estimates for the monotone SL scheme, we refer to [21, 8].)

2.4. Adding a limiter. The basic filter scheme (2.6) is designed to be of high-order where the solution is regular and when there is no viscosity aspects. However, for instance in the case of front propagation, it can be observed that the filter scheme may let small errors occur near extrema, when two possible directions of propagation occur in the same cell.

This is the case for instance near a minima for an eikonal equation. In order to improve the scheme near extrema, we propose to introduce a limiter before doing the filtering process. Limiting correction will be needed only when there is some viscosity aspect (it is not needed for advection).

Let us consider the case of front propagation, i.e., equation of type (2.4), now with

$$H(x, v_x) = \max_{a \in A} (f(x, a) v_x) \quad (2.32)$$

(i.e., no distributive cost in the Hamiltonian function).

In the one-dimensional case, a viscosity aspect may occur at a minima detected at mesh point x_i if

$$\min_a f(x_j, a) \leq 0 \quad \text{and} \quad \max_a f(x_j, a) \geq 0. \quad (2.33)$$

In that case, the solution should not go below the local minima around this point, i.e., we want

$$u_j^{n+1} \geq u_{min,j} := \min(u_{j-1}^n, u_j^n, u_{j+1}^n), \quad (2.34)$$

and, in the same way, we want to impose that

$$u_j^{n+1} \leq u_{max,j} := \max(u_{j-1}^n, u_j^n, u_{j+1}^n). \quad (2.35)$$

If we consider the high-order scheme to be of the form $u_j^{n+1} = u_j^n - \tau h^A(u^n)$, then the limiting process amounts to saying that

$$h^A(u^n)_j \leq h_j^{max} := \frac{u_j^n - u_{min,j}}{\tau}.$$

and

$$h^A(u^n)_j \geq h_j^{min} := \frac{u_j^n - u_{max,j}}{\tau}.$$

This amounts to define a limited \bar{h}^A such that, if (2.33) holds at mesh point x_j , then

$$\bar{h}^A(u^n)_j := \min \left(\max(h^A(u^n)_j, h_j^{min}), h_j^{max} \right).$$

and, otherwise,

$$\bar{h}_j^A \equiv h_j^A.$$

Then the filtering process is the same, using \bar{h}^A instead of h^A for the definition of the high-order scheme S^F .

For two dimensional equations a similar limiter could be developped in order to make the scheme more efficient at singular regions. However, for the numerical tests of the next section (in two dimensions) we will simply limit the scheme by using an equivalent of (2.34)-(2.35). Hence, instead of the scheme value $u_{ij}^{n+1} = S^A(u^n)_{ij}$ for the high-order scheme, we will update the value by

$$u_{ij}^{n+1} = \min(\max(S^A(u^n)_{ij}, u_{ij}^{min}), u_{ij}^{max}), \quad (2.36)$$

where $u_{ij}^{min} = \min(u_{ij}^n, u_{i\pm 1, j}^n, u_{i, j\pm 1}^n)$ and $u_{ij}^{max} = \max(u_{ij}^n, u_{i\pm 1, j}^n, u_{i, j\pm 1}^n)$.

2.5. How to choose the parameter ϵ : a simplified approach. The scheme should switch to high-order scheme when some regularity of the data is detected, and in that case we should have

$$\left| \frac{S^A(v) - S^M(v)}{\epsilon \tau} \right| = \left| \frac{h^A(\cdot) - h^M(\cdot)}{\epsilon} \right| \leq 1.$$

In a region where a function $v = v(x)$ is regular enough, by using Taylor expansions, zero order terms in $h^A(x, D^\pm v)$ and $h^M(x, D^\pm v)$ vanish (they are both equal to $H(x, v_x(x))$) and it remains an estimate of order Δx . More precisely, by using the high-order property (A4) we have

$$h^A(x_j, D^\pm v_j) = H(x_j, v_x(x_j)) + O(\Delta x^2).$$

On the other hand, by using Taylor expansions,

$$Dv_j^\pm = v_x(x_j) \pm \frac{1}{2} v_{xx}(x_j) \Delta x + O(\Delta x^2),$$

Hence, denoting $h^M = h^M(x, u^-, u^+)$, it holds at points where h^M is regular,

$$h^M(x_j, Dv_j^-, Dv_j^+) = H(x_j, v_x(x_j)) + \frac{1}{2} v_{xx}(x_j) \left(\frac{\partial h_j^M}{\partial u^+} - \frac{\partial h_j^M}{\partial u^-} \right) + O(\Delta x^2).$$

Therefore,

$$|h^A(v) - h^M(v)| = \frac{1}{2} |v_{xx}(x_j)| \left| \frac{\partial h_j^M}{\partial u^+} - \frac{\partial h_j^M}{\partial u^-} \right| \Delta x + O(\Delta x^2).$$

Hence we will make the choice to take ϵ roughly such that

$$\frac{1}{2} |v_{xx}(x_j)| \left| \frac{\partial h_j^M}{\partial u^+} - \frac{\partial h_j^M}{\partial u^-} \right| \Delta x \leq \epsilon \quad (2.37)$$

(where $h_j^M = h^M(x_j, v_x(x_j), v_x(x_j))$). Therefore, if at some point x_j (2.37) holds, then the scheme will switch to the high-order scheme. Otherwise, when the expectations from h^M and h^A are different enough, the scheme will switch to the monotone scheme.

In conclusion we have upper and lower bound for the switching parameter ϵ :

- Choose $\epsilon \leq c_0 \sqrt{\Delta x}$ for some constant $c_0 > 0$ in order that the convergence and error estimate result holds (see Theorem 2.1).

- Choose $\epsilon \geq c_1 \Delta x$, where c_1 is sufficiently large. This constant should be chosen roughly such that

$$\frac{1}{2} \|v_{xx}\|_{\infty} \left\| \frac{\partial h^M}{\partial u^+}(\cdot, v_x, v_x) - \frac{\partial h^M}{\partial u^-}(\cdot, v_x, v_x) \right\|_{\infty} \leq c_1.$$

where the range of values of v_x and v_{xx} can be estimated, in general, from the values of $(v_0)_x$, $(v_0)_{xx}$ and the Hamiltonian function H . Then the scheme is expected to switch to the high-order scheme where the solution is regular.

3. Numerical tests. In this section we present several numerical tests in one and two dimensions. Unless otherwise indicated, the filtered scheme will refer to the scheme where the high-order Hamiltonian is the centered scheme in space (see Remark 2.2), with Heun (RK2) scheme discretization in time (see in particular Eqs. (2.21a)-(2.21b)). Hereafter this scheme will be referred as the “centered scheme”.

The monotone finite difference scheme and function h^M will be made precise for each example.

For the filtered scheme, unless otherwise precised, the switching coefficient $\epsilon = 5\Delta x$ will be used. In practice we have numerically observed that taking $\epsilon = c_1 \Delta x$ with c_1 sufficiently large does not much change the numerical results in the following tests. All the tested filtered schemes (apart from the steady and obstacle equations) enters in the convergence framework of the previous section, so in particular there is a theoretical convergence of order $\sqrt{\Delta x}$ under the usual CFL condition.

In the tests, the filtered scheme will be in general compared to a second order ENO scheme (for precise definition, see Appendix A), as well as the centered (a priori unstable) scheme without filtering.

In several cases, local error in the L^2 norms are computed in some subdomain D , which, at a given time t_n , corresponds to

$$e_{L^2_{loc}} := \left(\sum_{\{i, x_i \in D\}} |v(t_n, x_i) - u_i^n|^2 \right)^{1/2}$$

The first two numerical examples deal with one-dimensional HJ equations, examples 3 and 4 are concerned with two-dimensional HJ equations, and the last three examples will concern a one-dimensional steady equation and two nonlinear one-dimensional obstacle problems.

Example 1. Eikonal equation. We consider the case of

$$v_t + |v_x| = 0, \quad t \in (0, T), \quad x \in (-2, 2), \quad (3.1)$$

$$v(0, x) = v_0(x) := \max(0, 1 - x^2)^4, \quad x \in (-2, 2). \quad (3.2)$$

In Table 3.1, we compare the filtered scheme (with $\epsilon = 5\Delta x$) with the centered scheme and the ENO second order scheme, with CFL = 0.37 and terminal time $T = 0.3$. For the filtered scheme, the monotone hamiltonian used is $h^M(x, v^-, v^+) := \max(v^-, -v^+)$.

As expected, we observe that the centered scheme alone is unstable. On the other hand, the filtered and ENO schemes are numerically comparable in that case, and second order convergent (the results are similar for the L^1 and the L^∞ errors).

Then, in Table 3.2, we consider the same PDE but with the following reversed initial data:

$$\tilde{v}_0(x) := -\max(0, 1 - x^2)^4, \quad x \in (-2, 2). \quad (3.3)$$

In that case the centered scheme alone is unbounded. The filtered scheme (with $\epsilon = 5\Delta x$) is second order. However, here, the limiter correction as described in section (2.4) was needed in order to

		filtered ($\epsilon = 5\Delta x$)		centered		ENO2	
M	N	L^2 error	order	L^2 error	order	L^2 error	order
40	9	7.51E-03	-	1.18E-01	-	1.64E-02	-
80	17	3.36E-03	1.16	1.14E-01	0.06	4.38E-03	1.91
160	33	8.02E-04	2.07	1.13E-01	0.00	1.19E-03	1.87
320	65	1.80E-04	2.16	1.13E-01	0.00	3.22E-04	1.89
640	130	4.53E-05	1.99	1.13E-01	0.00	8.22E-05	1.97

TABLE 3.1

(Example 1 with initial data (3.2)) L^2 errors for filtered scheme, centered scheme, and ENO second order scheme

		filtered ($\epsilon = 5\Delta x$)		centered		ENO2	
M	N	error	order	error	order	error	order
40	9	1.27E-02	-	2.03E-02	-	2.60E-02	-
80	17	3.17E-03	2.00	8.96E-03	1.18	8.00E-03	1.70
160	33	7.90E-04	2.01	1.06E-02	-0.24	2.50E-03	1.68
320	65	1.97E-04	2.00	1.26E-01	-3.57	7.80E-04	1.68
640	130	4.92E-05	2.00	1.06E+02	-9.71	2.44E-04	1.67

TABLE 3.2

(Example 1 with initial data (3.3).) L^2 errors for filtered scheme, centered scheme, and ENO second order scheme.

get second order behavior. We also observe that the filtered scheme gives better results than the ENO scheme. (We have also numerically tested the ENO scheme with the same limiter correction but it does not improve the behavior of the ENO scheme alone).

In conclusion, this first example shows firstly, that the filtered scheme can stabilize an otherwise unstable scheme, and secondly that it can give the desired second order behavior.

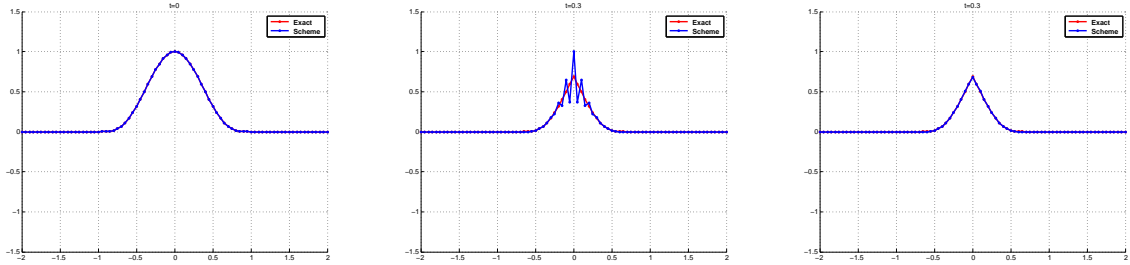


FIG. 3.1. (Example 1) With initial data (3.2) (left), and plots at time $T = 0.3$ with centered scheme - middle - and filtered scheme - right, using $M = 160$ mesh points.

Example 2. Burger's equation.

In this example an HJ equivalent of the nonlinear Burger's equation is considered:

$$v_t + \frac{1}{2}|v_x|^2 = 0, \quad t > 0, \quad x \in (-2, 2) \quad (3.4a)$$

$$v(0, x) = v_0(x) := \max(0, 1 - x^2), \quad x \in (-2, 2) \quad (3.4b)$$

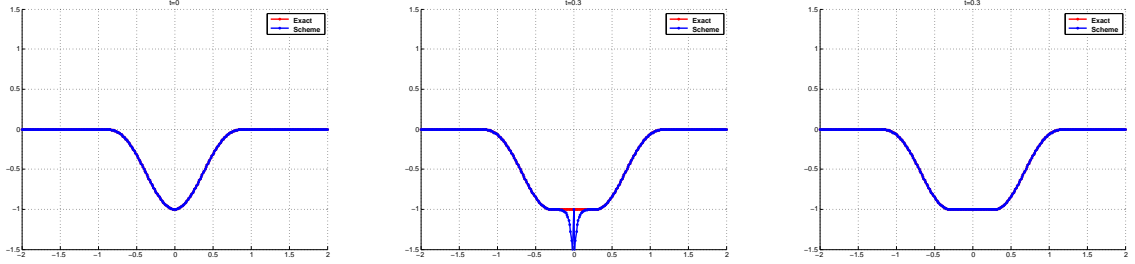


FIG. 3.2. (Example 1) With initial data (3.3) (left), and plots at time $T = 0.3$ with centered scheme - middle - and filtered scheme - right, using $M = 160$ mesh points.

		filtered ($\epsilon = 5\Delta x$)		centered		ENO2	
M	N	error	order	error	order	error	order
40	9	2.06E-02	-	2.07E-02	-	2.55E-02	-
80	17	6.24E-03	1.73	6.24E-03	1.73	8.24E-03	1.63
160	33	1.85E-03	1.76	1.85E-03	1.76	2.81E-03	1.55
320	65	5.51E-04	1.74	5.51E-04	1.74	1.03E-03	1.45
640	130	1.68E-04	1.71	1.68E-04	1.71	3.74E-04	1.47

TABLE 3.3

(Example 2) L^2 errors for filtered scheme, centered scheme, and ENO second order scheme.

with Dirichlet boundary condition on $(-2, 2)$. Exact solution is known.¹. In order to test high-order convergence we have considered the smoother initial data which is the one obtained from (3.4) at time $t_0 := 0.1$, i.e. :

$$w_t + \frac{1}{2}|w_x|^2 = 0, \quad t > 0, \quad x \in (-2, 2). \quad (3.5a)$$

$$w(0, x) := v(t_0, x), \quad x \in (-2, 2), \quad (3.5b)$$

with exact solution $w(t, x) = v(t + t_0, x)$.

An illustration is given in Fig. 3.3. For the filtered scheme, the monotone hamiltonian used is $h^M(x, v^-, v^+) := \frac{1}{2}(v^-)^2 1_{v^- > 0} + \frac{1}{2}(v^+)^2 1_{v^+ < 0}$.

Errors are given in Table (3.3), using CFL=0.37 and terminal time $T = 0.3$.

In conclusion we observe numerically that the filtered scheme keeps the good behavior of the centered scheme (here stable and almost second order).

Example 3. 2D rotation. We now apply filtered scheme to an advection equation in two

¹ It holds

$$v(t, x) = \frac{(\max(0, 1 - |x|))^2}{2t} 1_{\{t > \frac{1}{2}\}} + \frac{(1 - 2t)^2 - |x|^2}{1 - 2t} 1_{\{1 \geq |x| \geq 1 - 2t\}}.$$

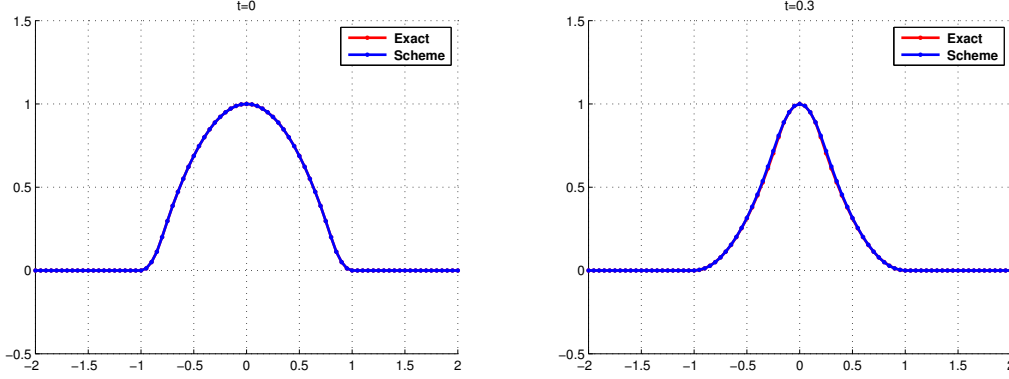


FIG. 3.3. (Example 2) Plots at $t = 0$ and $t = 0.3$ with the filtered scheme.

dimensions:

$$v_t - yv_x + xv_y = 0, \quad (x, y) \in \Omega, \quad t > 0, \quad (3.6)$$

$$v(0, x, y) = v_0(x, y) := 0.5 - 0.5 \max(0, \frac{1 - (x-1)^2 - y^2}{1 - r_0^2})^4 \quad (3.7)$$

where $\Omega := (-A, A)^2$ (with $A = 2.5$), $r_0 = 0.5$ and with Dirichlet boundary condition $v(t, x) = 0.5$, $x \in \partial\Omega$. This initial condition is regular and such that the level set $v_0(x, y) = 0$ corresponds to a circle centered at $(1, 0)$ and of radius r_0 .

In this example the monotone numerical Hamiltonian is defined by

$$h^M(u_x^-, u_x^+, u_y^-, u_y^+) := \max(0, f_1(a, x, y))u_x^- + \min(0, f_1(a, x, y))u_x^+ \\ + \max(0, f_2(a, x, y))u_y^- + \min(0, f_2(a, x, y))u_y^+ \quad (3.8)$$

and the high-order scheme is the centered finite difference scheme in both spacial variables, and RK2 in time. The filtered scheme is otherwise the same as (2.6). However it is necessary to use a greater constant c_1 is the choice $\epsilon = c_1 \Delta x$ in order to get (global) second order errors. We have used here $\epsilon = 20 \Delta x$.

On the other hand the CFL condition is

$$\mu := c_0 \left(\frac{\tau}{\Delta x} + \frac{\tau}{\Delta y} \right) \leq 1, \quad (3.9)$$

where here $c_0 = 2.5$ (an upper bound for the dynamics in the considered domain Ω). In this test the CFL number is $\mu := 0.37$.

Results are shown in Table 3.4 for terminal time $T := \pi/2$. Although the centered scheme is a priori unstable, in this example it is numerically stable and of second order. So we observe that the filtered scheme keep this good behavior and is also of second order (ENO scheme gives comparable results here).

Example 4. Eikonal equation. In this example we consider the eikonal equation

$$v_t + |\nabla v| = 0, \quad (x, y) \in \Omega, \quad t > 0 \quad (3.10)$$

in the domain $\Omega := (-3, 3)^2$. The initial data is defined by

$$v_0(x, y) = 0.5 - 0.5 \max \left(\max(0, \frac{1 - (x-1)^2 - y^2}{1 - r_0^2})^4, \max(0, \frac{1 - (x+1)^2 - y^2}{1 - r_0^2})^4 \right). \quad (3.11)$$

		filtered		centered		ENO	
Mx	Ny	L^2 error	order	L^2 error	order	L^2 error	order
20	20	5.05E-01	-	5.05E-01	-	6.99E-01	-
40	40	1.48E-01	1.77	1.48E-01	1.77	4.66E-01	0.58
80	80	3.77E-02	1.98	3.77E-02	1.98	2.04E-01	1.19
160	160	9.40E-03	2.00	9.40E-03	2.00	5.50E-02	1.89
320	320	2.34E-03	2.01	2.34E-03	2.01	1.29E-02	2.10

TABLE 3.4

(Example 3) Global L^2 errors for the filtered scheme, centered and second order ENO schemes (with CFL 0.37).

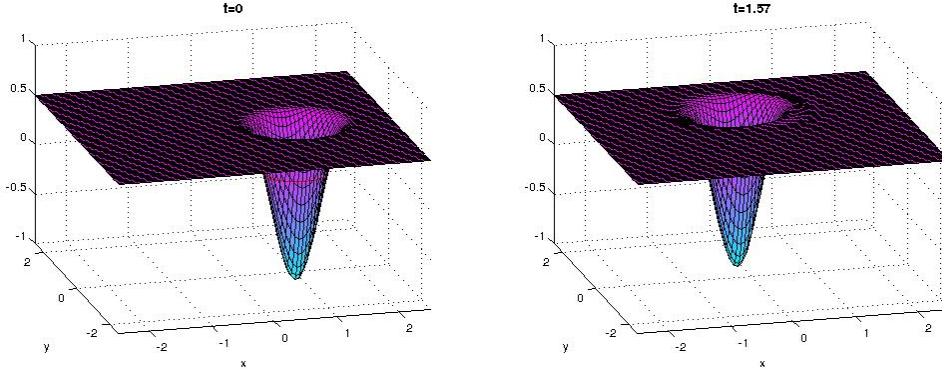


FIG. 3.4. (Example 3) Filtered scheme, plots at time $t = 0$ (left) and $t = \pi/2$ (right) with $M = 80$ mesh points.

The zero-level set of v_0 corresponds to two separates circles or radius r_0 and centered in $A = (1, 0)$ and $B = (-1, 0)$ respectively. Dirichlet boundary conditions are used as the previous example.

The monotone hamiltonian h^M used in the filtered scheme is in Lax-Friedrichs form:

$$h^M(x, u_1^-, u_1^+, u_2^-, u_2^+) = H(x, \frac{u_1^- + u_1^+}{2}, \frac{u_2^- + u_2^+}{2}) - \frac{C_x}{2}(u_1^+ - u_1^-) - \frac{C_y}{2}(u_2^+ - u_2^-), \quad (3.12)$$

where, here, $C_x = C_y = 1$. We used the CFL condition $\mu = 0.37$ as in (3.9). Also, the simple limiter (2.36) was used for the filtered scheme as described in Section 2.4. It is needed in order to get a good second order behavior at extrema of the numerical solution. The filter coefficient is set to $\epsilon = 20\Delta x$ as in the previous example.

Numerical results are given in Table 3.5, showing the global L^2 errors for the filtered scheme, the centered scheme, and a second order ENO scheme, at time $t = 0.6$. We observe that the centered scheme has some unstabilities for fine mesh, while the filtered performs as expected.

Example 5 Steady eikonal equation. We consider a steady eikonal equation with Dirichlet boundary condition, which is taken from Abgrall [1]:

$$|v_x| = f(x) \quad x \in (0, 1), \quad (3.13a)$$

$$v(0) = v(1) = 0, \quad (3.13b)$$

where $f(x) = 3x^2 + a$, with $a = \frac{1-2x_0^3}{2x_0-1}$ and $x_0 = \frac{\sqrt[3]{2}+2}{4\sqrt[3]{2}}$. Exact solution is known:

$$v(x) := \begin{cases} x^3 + ax & x \in [0, x_0], \\ 1 + a - ax - x^3 & x \in [x_0, 1]. \end{cases} \quad (3.14)$$

		filtered ($\epsilon = 20\Delta x$)		centered		ENO2	
Mx	Ny	L^2 error	order	L^2 error	order	L^2 error	order
25	25	5.39E-01	-	6.00E-01	-	5.84E-01	-
50	50	1.82E-01	1.57	2.25E-01	1.41	2.11E-01	1.47
100	100	3.72E-02	2.29	8.46E-02	1.41	6.88E-02	1.62
200	200	9.36E-03	1.99	3.53E-02	1.26	2.02E-02	1.76
400	400	2.36E-03	1.99	1.36E-01	-1.95	5.98E-03	1.76

TABLE 3.5

(Example 4) Global L^2 errors for filtered scheme, centered and second order ENO schemes.

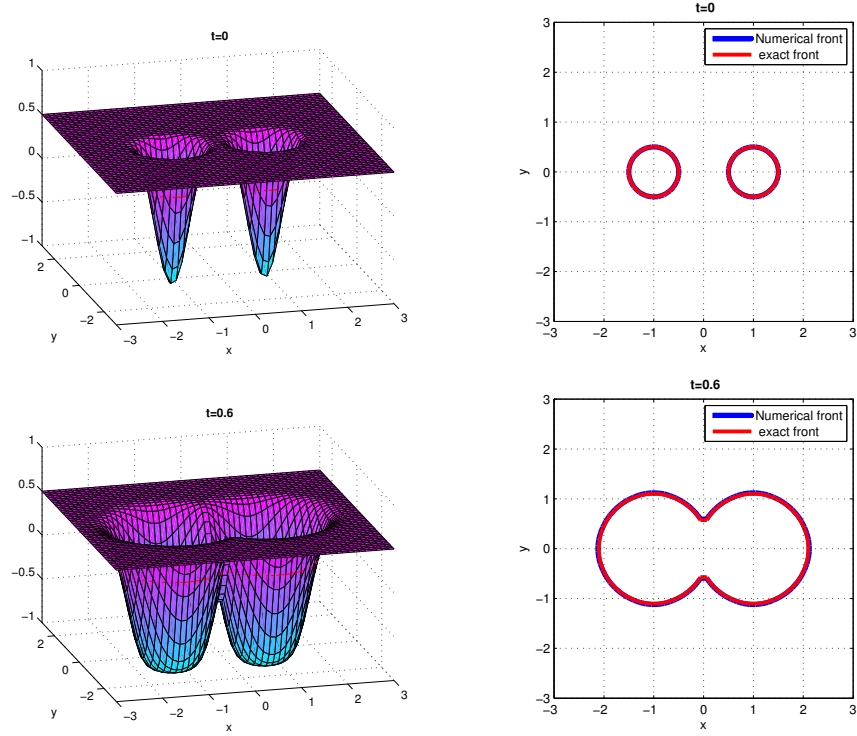


FIG. 3.5. (Example 4) Plots at times $t = 0$ (top) and $t = \pi/2$ (bottom) for the filtered scheme with $M = 50$ mesh points. The figures to the right represent the 0-level sets.

The steady solution for (3.13) can be considered as the limit $\lim_{t \rightarrow \infty} v(t, x)$ where v is the solution of the time marching corresponding form:

$$v_t + |v_x| = f(x) \quad x \in (0, 1), \quad t > 0, \quad (3.15a)$$

$$v(t, 0) = v(t, 1) = 0, \quad t > 0. \quad (3.15b)$$

In this example the upwind monotone scheme is used:

$$h^M(\cdot)_j := \frac{u_j^{n+1} - u_j^n}{\tau} - \max\left\{\frac{u_j^n - u_{j-1}^n}{\Delta x}, \frac{u_j^n - u_{j+1}^n}{\Delta x}\right\} - \tau f(x_j),$$

the high-order scheme will be the centered scheme, and the filtered scheme (2.6) will be used with $\epsilon = 5\Delta x$. The iterations are stopped when the difference between two successive time step is small

enough or a fixed number of iterations is passed, i.e., in this example,

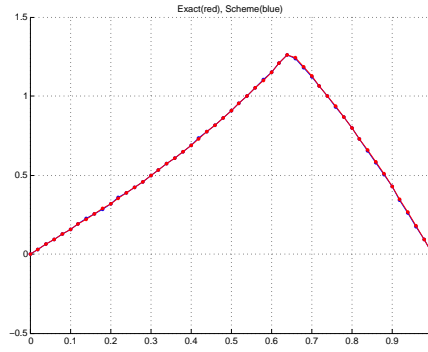
$$\|u^{n+1} - u^n\|_{L^\infty} := \max_i |u_i^{n+1} - u_i^n| \leq 10^{-6} \quad \text{or} \quad n \geq N_{max} := 5000. \quad (3.16)$$

As analyzed in [4] for ϵ -monotone schemes, for a given mesh step, even if the iterations may not converge (because of the non monotony of the scheme), it can be shown to be close to a fixed point after enough iterations.

	filtered		centered		filtered + ENO	
M	error	order	error	order	error	order
50	2.16E-03	-	NaN	-	5.29E-03	-
100	7.14E-04	1.60	NaN	-	1.35E-03	1.97
200	2.17E-04	1.72	NaN	-	3.42E-04	1.98
400	6.32E-05	1.78	NaN	-	8.61E-05	1.99
800	2.17E-05	1.54	NaN	-	2.16E-05	2.00

TABLE 3.6

(Example 5) Global errors for filtered scheme, compared with the centered (unstable) scheme, and a filtered ENO scheme.



Example 6 Advection with an obstacle. Here we consider an obstacle problem, which is taken from [3]:

$$\min(v_t + v_x, v - g(x)) = 0, \quad t > 0, x \in [-1, 1], \quad (3.17)$$

$$v_0(x) = 0.5 + \sin(\pi x), \quad x \in [-1, 1] \quad (3.18)$$

together with periodic boundary condition. The obstacle function is $g(x) := \sin(\pi x)$. In this case exact solution is given by:

$$v(t, x) := \begin{cases} \max(v_0(x - at), g(x)) & \text{if } t < \frac{1}{3} \\ \max(v_0(x - at), g(x), -1_{x \in [0.5, 1]}) & \text{if } t \in [\frac{1}{3}, \frac{5}{6}], \\ \max(v_0(x - at), g(x), 1_{x \in [-1, t - \frac{5}{6}] \cup [0.5, 1]}) & \text{if } t \in [\frac{5}{6}, 1], \end{cases} \quad (3.19)$$

Results are given in Table 3.7, for terminal time $t = 0.5$. Errors are computed away from singular points, i.e., in the region $[-1, 1] \setminus (\cup_{i=1,3} [s_i - \delta, s_i + \delta])$ (where $s_1 = -0.1349733, s_2 = 0.5$ and $s_3 = 2/3$ are the three singular points. Filtered scheme is numerically of second order (ENO gives comparable results here).

Errors		filtered $\epsilon = 5\Delta x$		centered		ENO2	
M	N	error	order	error	order	error	order
40	20	7.93E-03	2.03	1.63E-02	1.54	2.14E-02	1.59
80	40	1.84E-03	2.10	2.98E-02	-0.87	7.75E-03	1.46
160	80	3.92E-04	2.24	1.46E-02	1.03	1.07E-03	2.86
320	160	9.67E-05	2.02	8.02E-03	0.86	2.72E-04	1.97
640	320	2.40E-05	2.01	4.10E-03	0.97	6.92E-05	1.98

TABLE 3.7

(Example 6) L^∞ errors away from singular points, for filtered scheme, centered scheme, and second order ENO scheme.

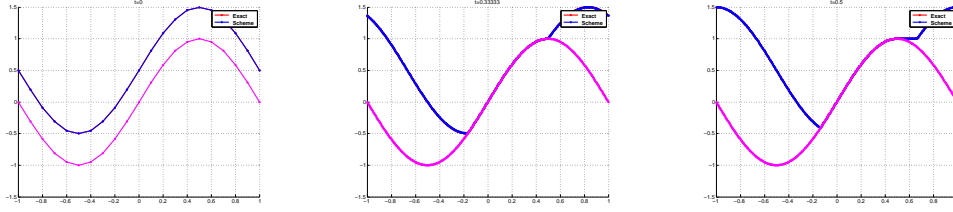


FIG. 3.7. (Example 6) Plots at $T=0$ (initial data), $T=0.3$, $T=0.5$.

Example 7 Eikonal with an obstacle. We consider an Eikonal equation with an obstacle term, also taken from [3]:

$$\min(v_t + |v_x|, v - g(x)) = 0, \quad t > 0, x \in [-1, 1], \quad (3.20)$$

$$v_0(x) = 0.5 + \sin(\pi x) \quad x \in [-1, 1], \quad (3.21)$$

with periodic boundary condition on $(-1, 1)$ and $g(x) = \sin(\pi x)$. In this case the exact solution is given by:

$$v(t, x) = \max(\bar{v}(t, x), g(x)). \quad (3.22)$$

where \bar{v} is the solution of the Eikonal equation $v_t + |v_x| = 0$. The formula $\bar{v}(t, x) = \min_{y \in [x-t, x+t]} v_0(y)$ holds, which simplifies to

$$v(t, x) := \begin{cases} v_0(x+t) & \text{if } x < -0.5 - t \\ -0.5 & \text{if } x \in [-0.5 - t, -0.5 + t], \\ \min(v_0(x-t), v_0(x+t)) & \text{if } x \geq -0.5 + t, \end{cases} \quad (3.23)$$

Results are given in Table 3.8 for terminal time $T = 0.2$. Plots are also shown in Figure 3.8 for different times (for $t \geq \frac{1}{3}$ solution remains unchanged).

4. Conclusion. We propose a “filtered” scheme which behaves as a high-order scheme when the solution is smooth and as a low order monotone scheme otherwise. It has a simple presentation that is easy to implement. Rigorous error bounds hold, of the same order as the Crandall-Lions estimates in $\sqrt{\Delta x}$ where Δx is the mesh size. In the case the solution is smooth a high-order consistency error estimate also holds. We show on several numerical examples the ability of the scheme to stabilize an otherwise unstable scheme, and also we observe a precision similar to a second order ENO scheme on basic linear and non linear examples.

Errors	filtered		ENO2	
M	error	order	error	order
40	3.74E-03	-	6.85E-03	-
80	6.26E-04	2.58	2.12E-03	1.69
160	1.13E-04	2.47	6.80E-04	1.64
320	2.26E-05	2.32	2.18E-04	1.64
640	5.50E-06	2.04	6.96E-05	1.65

TABLE 3.8
Filtered scheme and ENO scheme at time $t = 0.2$

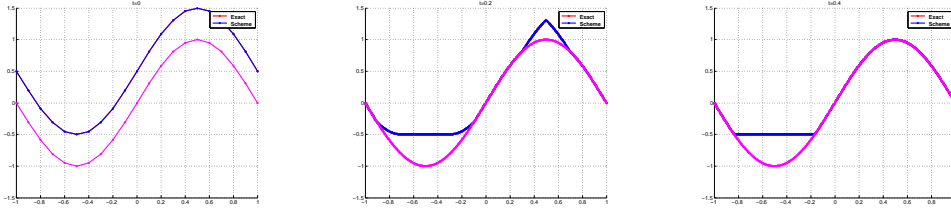


FIG. 3.8. (Example 7) Plots at times $t = 0$, $t = 0.2$ and $t = 0.4$. The dark line is the numerical solution, similar to the exact solution, and the light line is the obstacle function.

On going works concern the application of the present approach to some front propagation equations.

Appendix A. An essentially non-oscillatory (ENO) scheme of second order.

We recall here a simple ENO method of order two based on the work of Osher and Shu [20] for Hamilton Jacobi equation (the ENO method was designed by Harten et al. [17] for the approximation solution of non-linear conservation law).

Let m be the minmod function defined by

$$m(a, b) = \begin{cases} a & \text{if } |a| \leq |b|, \quad ab > 0 \\ b & \text{if } |b| < |a|, \quad ab > 0 \\ 0 & \text{if } ab \leq 0 \end{cases} \quad (\text{A.1})$$

(other functions can be considered such as $m(a, b) = a$ if $|a| \leq |b|$ and $m(a, b) = b$ otherwise). Let $D^\pm u_j = \pm(u_{j\pm 1} - u_j)/\Delta x$ and

$$D^2 u_j := \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}.$$

Then the right and left ENO approximation of the derivative can be defined by

$$\bar{D}^\pm u_j = D^\pm u_j \mp \frac{1}{2} \Delta x \, m(D^2 u_j, D^2 u_{j\pm 1})$$

and the ENO (Euler forward) scheme by

$$S_0(u)_j := u_j - \tau h^M(x_j, \bar{D}^- u_j, \bar{D}^+ u_j).$$

The corresponding RK2 scheme can then be defined by $S(u) = \frac{1}{2}(u + S_0(S_0(u)))$.

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